

A SYMMETRIC 2-TENSOR CANONICALLY ASSOCIATED TO Q -CURVATURE AND ITS APPLICATIONS

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ABSTRACT. In this article, we define a symmetric 2-tensor canonically associated to Q -curvature called J -tensor on any Riemannian manifold with dimension at least three. The relation between J -tensor and Q -curvature is precisely like Ricci tensor and scalar curvature. Thus it can be interpreted as a higher-order analogue of Ricci tensor. This tensor can also be used to understand Chang-Gursky-Yang's theorem on 4-dimensional Q -singular metrics. Moreover, we show an *Almost-Schur Lemma* holds for Q -curvature, which gives an estimate of Q -curvature on closed manifolds.

1. INTRODUCTION

Let M be a smooth manifold and \mathcal{M} be the space of all metrics on M . Consider scalar curvature as a nonlinear map

$$R : \mathcal{M} \rightarrow C^\infty(M); \quad g \mapsto R_g.$$

It is well-known that the linearization of scalar curvature at a given metric g (see [1, 6, 8]) is

$$(1.1) \quad \gamma_g h := DR_g \cdot h = -\Delta_g \operatorname{tr}_g h + \delta_g^2 h - \operatorname{Ric}_g \cdot h,$$

where $h \in S_2(M)$ is a symmetric 2-tensor and $\delta_g = -\operatorname{div}_g$. Thus, its L^2 -formal adjoint is given by

$$(1.2) \quad \gamma_g^* f = \nabla_g^2 f - g \Delta_g f - f \operatorname{Ric}_g,$$

for any smooth function $f \in C^\infty(M)$.

An interesting observation is that, if we take f to be constantly 1, we get

$$\operatorname{Ric}_g = -\gamma_g^* 1$$

That means we can recover Ricci tensor from γ_g^* . Furthermore, the scalar curvature is given by

$$R_g = -\operatorname{tr}_g \gamma_g^* 1.$$

Now let (M^n, g) be an n -dimensional Riemannian manifold ($n \geq 3$). We can define the Q -curvature to be

$$(1.3) \quad Q_g = A_n \Delta_g R_g + B_n |\operatorname{Ric}_g|_g^2 + C_n R_g^2,$$

where $A_n = -\frac{1}{2(n-1)}$, $B_n = -\frac{2}{(n-2)^2}$ and $C_n = \frac{n^2(n-4)+16(n-1)}{8(n-1)^2(n-2)^2}$.

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In fact, Q -curvature was introduced originally to generalize the classic *Gauss-Bonnet Theorem* on surfaces to closed 4-manifolds (M^4, g) :

$$(1.4) \quad \int_{M^4} \left(Q_g + \frac{1}{4} |W_g|_g^2 \right) dv_g = 8\pi^2 \chi(M).$$

where W_g is the Weyl tensor.

Paneitz and Branson extended it to any dimension $n \geq 3$ (cf. [2, 13]) such that it satisfies certain conformal invariant properties. For more details, please refer to the appendix of [12].

Like the scalar curvature, we can also view Q -curvature as a nonlinear map

$$Q : \mathcal{M} \rightarrow C^\infty(M); \quad g \mapsto Q_g.$$

Let

$$\Gamma_g : S_2(M) \rightarrow C^\infty(M)$$

be the linearization of Q -curvature at the metric g and

$$\Gamma_g^* : C^\infty(M) \rightarrow S_2(M)$$

be its L^2 -formal adjoint.

Now we can define the central notion in this article:

Definition 1.1. Let (M^n, g) be a Riemannian manifold ($n \geq 3$). We define the symmetric 2-tensor

$$J_g := -\frac{1}{2} \Gamma_g^* 1.$$

We say (M, g) is *J-Einstein*, if $J_g = \Lambda g$ for some smooth function $\Lambda \in C^\infty(M)$. In particular, it is *J-flat*, if $\Lambda = 0$.

In [12], we calculated the explicit expression of Γ_g^* and showed that

$$(1.5) \quad tr_g \Gamma_g^* f = \frac{1}{2} \left(P_g - \frac{n+4}{2} Q_g \right) f,$$

for any $f \in C^\infty(M)$. Here P_g is the *Paneitz operator* defined by

$$(1.6) \quad P_g = \Delta_g^2 - div_g [(a_n R_g g + b_n Ric_g) d] + \frac{n-4}{2} Q_g,$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$ and $b_n = -\frac{4}{n-2}$.

In particular,

$$tr_g \Gamma_g^* 1 = -2Q_g.$$

Thus

$$(1.7) \quad tr_g J_g = Q_g.$$

On the other hand, for any smooth vector field $X \in \mathcal{X}(M)$ on M ,

$$\int_M \langle X, \delta_g \Gamma_g^* f \rangle dv_g = \frac{1}{2} \int_M \langle L_X g, \Gamma_g^* f \rangle dv_g = \frac{1}{2} \int_M f \Gamma_g(L_X g) dv_g = \frac{1}{2} \int_M \langle f dQ_g, X \rangle dv_g.$$

Thus

$$\delta_g \Gamma_g^* f = \frac{1}{2} f dQ_g$$

on M . Hence,

$$(1.8) \quad \operatorname{div}_g J_g = \frac{1}{2} \delta_g \Gamma_g^* 1 = \frac{1}{4} dQ_g.$$

Recall that for Ricci tensor, we have

$$\operatorname{tr}_g \operatorname{Ric}_g = R_g$$

and

$$\operatorname{div}_g \operatorname{Ric}_g = \frac{1}{2} dR_g.$$

Therefore, if we consider Q -curvature as a higher-order analogue of scalar curvature, we can interpret J_g as a higher-order analogue of Ricci curvature on Riemannian manifolds.

A notion closely related to J -tensor is the Q -singular metric, which refers to a metric satisfying $\ker \Gamma_g^* \neq \{0\}$. Clearly, J -flat metrics are Q -singular, since it is equivalent to $1 \in \ker \Gamma_g^*$.

One of the motivations for us to study the J -flat manifold is to understand the following theorem by Chang-Gursky-Yang:

Theorem 1.2 (Chang-Gursky-Yang [4]). *Let (M^4, g) be a Q -singular 4-manifold. Then $1 \in \ker \Gamma_g^*$ if and only if (M^4, g) is Bach flat with vanishing Q -curvature.*

To achieve our goal, we need to give the explicit expression of J -tensor:

Theorem 1.3. *For $n \geq 3$,*

$$(1.9) \quad J_g = \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g,$$

where B_g is the Bach tensor and

$$T_g := (n-2) \left(\nabla^2 \operatorname{tr}_g S_g - \frac{1}{n} g \Delta_g \operatorname{tr}_g S_g \right) + 4(n-1) \left(S_g \times S_g - \frac{1}{n} |S_g|^2 g \right) - n^2 (\operatorname{tr}_g S_g) \mathring{S}_g.$$

Here $(S \times S)_{jk} = S_j^i S_{ik}$, S_g is the Schouten tensor and \mathring{S}_g is its traceless part.

Remark 1.4. Note that both the Bach tensor and the tensor T are traceless, thus the traceless part of J is given by

$$(1.10) \quad \mathring{J}_g = J_g - \frac{1}{n} Q_g g = -\frac{1}{n-2} \left(B_g + \frac{n-4}{4(n-1)} T_g \right).$$

Thus, an equivalent definition for a metric g being J -Einstein is

$$(1.11) \quad B_g = -\frac{n-4}{4(n-1)}T_g.$$

In particular, when $n = 4$, J -Einstein metrics are exactly Bach flat ones. Hence we can also interpret that J -Einstein metric is a generalization of Bach flat metric on 4-dimensional manifolds.

Remark 1.5. Gursky introduced a similar tensor for 4-manifolds from the viewpoint of functional determinant in [11]. In the same article, he also remarked this tensor can be introduced from the perspective of first variations of total Q -curvature when dimension is at least 5 (see [3] for a detailed calculation by Case).

With the similar perspective, Gover and Ørsted introduced an abstract tensor called *higher Einstein tensor*, which coincides with our J -tensor in one of its special case. We refer their article [10] for readers who are interested in it.

Note that for any Einstein metric g , its Q -curvature is given by

$$Q_g = B_n |Ric_g|^2 + C_n R_g^2 = \left(\frac{1}{n} B_n + C_n \right) R_g^2 = \frac{(n+2)(n-2)}{8n(n-1)^2} R_g^2,$$

which is a nonnegative constant and vanishes if and only if g is Ricci flat.

It is easy to check that $T_g = 0$ for any Einstein metric g . Combining this with the well-known fact that any Einstein metric is Bach flat, we can easily deduce that any non-flat Einstein metrics are also positive J -Einstein and Ricci flat metrics are J -flat as well.

With the aid of this notion, we can recover and generalize Theorem 1.2 to any dimension $n \geq 3$:

Corollary 1.6. *Let (M^n, g) be a Q -singular n -dimensional Riemannian manifold. Then $1 \in \ker \Gamma_g^*$ if and only if (M^n, g) is J -flat or equivalently (M^n, g) satisfies*

$$B_g = -\frac{n-4}{4(n-1)}T_g$$

with vanishing Q -curvature.

Remark 1.7. In [4], Bach flatness in Theorem 1.2 is derived using the variational property of Bach tensor for 4-manifolds.

As another application of J -tensor, we can derive the *Schur Lemma for Q -curvature* as follows:

Theorem 1.8 (Schur lemma). *Let (M^n, g) be an n -dimensional J -Einstein manifold with $n \neq 4$ or equivalently,*

$$B_g = -\frac{n-4}{4(n-1)}T_g,$$

then Q_g is a constant on M .

Moreover, the following *Almost-Schur Lemma* holds exactly like the case for Ricci tensor and scalar curvature (cf. [5, 7, 9]).

Theorem 1.9 (Almost-Schur Lemma). *For $n \neq 4$, let (M^n, g) be an n -dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$(1.12) \quad \int_M (Q_g - \overline{Q}_g)^2 dv_g \leq \frac{16n(n-1)}{(n-4)^2} \int_M |\overset{\circ}{J}_g|^2 dv_g,$$

where \overline{Q}_g is the average of Q_g . Moreover, the equality holds if and only if (M, g) is J -Einstein.

In order to derive an equivalent form of above inequality, we need to define the J -Schouten tensor as follows,

$$(1.13) \quad S_J = \frac{1}{n-4} \left(J_g - \frac{3}{4(n-1)} Q_g g \right).$$

Immediately, we have

$$(1.14) \quad tr_g S_J = \frac{1}{4(n-1)} Q_g$$

and

$$(1.15) \quad div_g S_J = \frac{1}{4(n-1)} dQ_g = dtr_g S_J.$$

Remark 1.10. Recall the definition of classic Schouten tensor

$$(1.16) \quad S_g = \frac{1}{n-2} \left(Ric_g - \frac{1}{2(n-1)} R_g g \right),$$

we have

$$(1.17) \quad tr_g S_g = \frac{1}{2(n-1)} R_g$$

and

$$(1.18) \quad div_g S_g = \frac{1}{2(n-1)} dR_g = dtr_g S_g.$$

We can see that the tensor S_J shares similar properties with the classic Schouten tensor.

Following the observation in [9], we get immediately the following result by rewriting the Theorem 1.9 with J -Schouten tensor:

Corollary 1.11. *For $n \neq 4$, let (M^n, g) be an n -dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$(1.19) \quad (Vol_g(M))^{-\frac{n-8}{n}} \int_M \sigma_2^J(g) dv_g \leq \frac{n-1}{2n} Y_Q^2(g),$$

where

$$Y_Q(g) := \frac{\int_M \sigma_1^J(g) dv_g}{(Vol_g(M))^{\frac{n-4}{n}}}$$

is the Q -Yamabe quotient and $\sigma_i^J(g) = \sigma_i(S_J(g))$, $i = 1, 2$ are the i^{th} -symmetric polynomial of $S_J(g)$. Moreover, the equality holds if and only if (M, g) is J -Einstein.

Remark 1.12. The above *Almost Schur Lemma* can be easily generalized to a broader setting by combining the work [10]. More detailed discussions together with some related topics will be presented in a subsequent article coming later.

This article is organized as follows: in Section 2, we derived the explicit formula for J -tensor and with the aid of it we proved Theorem 1.3 and Corollary 1.6; We then proved Theorem 1.8 (Schur Lemma) and Theorem 1.9 (Almost-Schur Lemma) in Section 3.

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2. J -FLATNESS AND Q -SINGULAR METRICS

We begin with some discussions of conformal tensors.

Let

$$(2.1) \quad S_{jk} = \frac{1}{n-2} \left(R_{jk} - \frac{1}{2(n-1)} R g_{jk} \right)$$

be the Schouten tensor.

For $n \geq 4$, the Bach tensor is defined to be

$$(2.2) \quad B_{jk} = \frac{1}{n-3} \nabla^i \nabla^l W_{ijkl} + W_{ijkl} S^{il}.$$

In order to extend the definition to $n = 3$, we introduce the Cotton tensor as follows

$$(2.3) \quad C_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik}$$

and it is related to Weyl tensor by the equation

$$(2.4) \quad \nabla^l W_{ijkl} = (n-3) C_{ijk}.$$

Therefore, for any $n \geq 3$, we can defined the Bach tensor as

$$(2.5) \quad B_{jk} = \nabla^i C_{ijk} + W_{ijkl} S^{il}.$$

The following identity is well-known for experts, we include calculations here for the convenience of readers.

Proposition 2.1. The Bach tensor can be written as

$$(2.6) \quad B_g = \Delta_g S - \nabla^2 \text{tr} S + 2\overset{\circ}{R}m \cdot S - (n-4)S \times S - |S|^2 g - 2(\text{tr} S)S,$$

where $(\overset{\circ}{R}m \cdot S)_{jk} = R_{ijkl}S^{il}$ and $(S \times S)_{jk} = S_j^i S_{ik}$. Equivalently,

$$(2.7) \quad B_g = \Delta_L S - \nabla^2 \text{tr} S + n \left(S \times S - \frac{1}{n} |S|^2 g \right),$$

where Δ_L is the Lichnerowicz Laplacian.

Proof. By the *second contracted Bianchi identity*,

$$\begin{aligned} \nabla^i S_{ik} &= \frac{1}{n-2} \left(\nabla^i R_{ik} - \frac{1}{2(n-1)} \nabla_k R \right) \\ &= \frac{1}{n-2} \left(\frac{1}{2} \nabla_k R - \frac{1}{2(n-1)} \nabla_k R \right) \\ &= \frac{1}{2(n-1)} \nabla_k R \\ &= \nabla_k \text{tr} S \end{aligned}$$

and

$$\text{tr} S = \frac{1}{n-2} \left(R - \frac{n}{2(n-1)} R \right) = \frac{1}{2(n-1)} R,$$

we have

$$\text{Ric} = (n-2)S + (\text{tr} S)g.$$

Using these facts,

$$\begin{aligned} \nabla^i C_{ijk} &= \nabla^i (\nabla_i S_{jk} - \nabla_j S_{ik}) \\ &= \Delta_g S_{jk} - (\nabla_j \nabla_i S_k^i + R_{ijp}^i S_k^p - R_{ijk}^p S_p^i) \\ &= \Delta_g S_{jk} - \nabla_j \nabla_k \text{tr} S - (\text{Ric} \times S)_{jk} + (\overset{\circ}{R}m \cdot S)_{jk} \\ &= \Delta_g S_{jk} - \nabla_j \nabla_k \text{tr} S - ((n-2)S + (\text{tr} S)g) \times S_{jk} + (\overset{\circ}{R}m \cdot S)_{jk} \\ &= \Delta_g S_{jk} - \nabla_j \nabla_k \text{tr} S - (n-2)(S \times S)_{jk} - (\text{tr} S)S_{jk} + (\overset{\circ}{R}m \cdot S)_{jk} \end{aligned}$$

and

$$\begin{aligned} W_{ijkl}S^{il} &= (Rm - S \oslash g)_{ijkl} S^{il} \\ &= R_{ijkl}S^{il} - (S_{il}g_{jk} + S_{jk}g_{il} - S_{ik}g_{jl} - S_{jl}g_{ik})S^{il} \\ &= (\overset{\circ}{R}m \cdot S)_{jk} - |S|^2 g_{jk} + 2(S \times S)_{jk} - (\text{tr} S)S_{jk}, \end{aligned}$$

where \oslash is the *Kulkarni-Nomizu product*:

$$(\alpha \oslash \beta)_{ijkl} := \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{ik}\beta_{jl} - \alpha_{jl}\beta_{ik}$$

for any symmetric 2-tensor $\alpha, \beta \in S_2(M)$.

Combine them, we get

$$B_{jk} = \Delta_g S_{jk} - \nabla_j \nabla_k \text{tr} S + 2(\overset{\circ}{R}m \cdot S)_{jk} - (n-4)(S \times S)_{jk} - |S|^2 g_{jk} - 2(\text{tr} S) S_{jk}.$$

From this,

$$\begin{aligned} B_{jk} &= \Delta_L S_{jk} + 2(\text{Ric} \times S)_{jk} - \nabla_j \nabla_k \text{tr} S - (n-4)(S \times S)_{jk} - |S|^2 g_{jk} - 2(\text{tr} S) S_{jk} \\ &= \Delta_L S_{jk} + 2((\text{Ric} - (\text{tr} S)g) \times S)_{jk} - \nabla_j \nabla_k \text{tr} S - (n-4)(S \times S)_{jk} - |S|^2 g_{jk} \\ &= \Delta_L S - \nabla^2 \text{tr} S + n(S \times S) - |S|^2 g \\ &= \Delta_L S - \nabla^2 \text{tr} S + n \left(S \times S - \frac{1}{n} |S|^2 g \right). \end{aligned}$$

□

The Q -curvature can also be rewritten using Schouten tensor:

Lemma 2.2.

$$(2.8) \quad Q_g = -\Delta_g \text{tr} S - 2|S|^2 + \frac{n}{2}(\text{tr} S)^2.$$

Proof. Using the equalities $\text{Ric} = (n-2)S + (\text{tr} S)g$ and $R = 2(n-1)\text{tr} S$,

$$\begin{aligned} Q_g &= A_n \Delta_g R + B_n |\text{Ric}|^2 + C_n R^2 \\ &= 2(n-1)A_n \Delta_g \text{tr} S + B_n |(n-2)S + (\text{tr} S)g|^2 + 4(n-1)^2 C_n (\text{tr} S)^2 \\ &= -\Delta_g \text{tr} S - 2|S|^2 + ((3n-4)B_n + 4(n-1)^2 C_n)(\text{tr} S)^2 \\ &= -\Delta_g \text{tr} S - 2|S|^2 + \frac{n}{2}(\text{tr} S)^2. \end{aligned}$$

□

We recall the expression of Γ_g^* in [12] as follows:

Lemma 2.3.

$$\begin{aligned} (2.9) \quad \Gamma_g^* f &:= A_n \left(-g \Delta^2 f + \nabla^2 \Delta f - \text{Ric} \Delta f + \frac{1}{2} g \delta(f dR) + \nabla(f dR) - f \nabla^2 R \right) \\ &\quad - B_n \left(\Delta(f \text{Ric}) + 2f \overset{\circ}{R}m \cdot \text{Ric} + g \delta^2(f \text{Ric}) + 2\nabla \delta(f \text{Ric}) \right) \\ &\quad - 2C_n (g \Delta(f R) - \nabla^2(f R) + f R \text{Ric}). \end{aligned}$$

Now we can calculate the explicit expression of J_g :

Theorem 2.4. For $n \geq 3$,

$$(2.10) \quad J_g = \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g,$$

where

$$T_g := (n-2) \left(\nabla^2 \text{tr}_g S_g - \frac{1}{n} g \Delta_g \text{tr}_g S_g \right) + 4(n-1) \left(S_g \times S_g - \frac{1}{n} |S_g|^2 g \right) - n^2 (\text{tr}_g S_g) \overset{\circ}{S}_g.$$

Here $\overset{\circ}{S}_g = S_g - \frac{1}{n} \text{tr}_g S_g g$ is the traceless part of Schouten tensor.

Proof. By Lemma 2.3,

$$\begin{aligned} \Gamma_g^* 1 &= - \left(\frac{1}{2} A_n + \frac{1}{2} B_n + 2C_n \right) g \Delta R + (B_n + 2C_n) \nabla^2 R \\ &\quad - B_n (\Delta \text{Ric} + 2 \overset{\circ}{R} m \cdot \text{Ric}) - 2C_n R \text{Ric}. \end{aligned}$$

Applying equalities $\text{Ric} = (n-2)S + (\text{tr} S)g$ and $R = 2(n-1)\text{tr} S$,

$$\begin{aligned} \Gamma_g^* 1 &= - ((n-1)A_n + nB_n + 4(n-1)C_n) g \Delta \text{tr} S + 2(n-1)(B_n + 2C_n) \nabla^2 \text{tr} S \\ &\quad - (n-2)B_n (\Delta S + 2 \overset{\circ}{R} m \cdot S) - 2(n-2)(B_n + 2(n-1)C_n) (\text{tr} S) S \\ &\quad - 2(B_n + 2(n-1)C_n) (\text{tr} S)^2 g \\ &= \frac{3}{2(n-1)} g \Delta \text{tr} S + \frac{2}{n-2} (\Delta S + 2 \overset{\circ}{R} m \cdot S) + \frac{n^2 - 10n + 12}{2(n-1)(n-2)} \nabla^2 \text{tr} S \\ &\quad - \frac{n^2 - 2n + 4}{2(n-1)} (\text{tr} S) S - \frac{n^2 - 2n + 4}{2(n-1)(n-2)} (\text{tr} S)^2 g. \end{aligned}$$

Since $\text{tr} \Gamma_g^* 1 = -2Q_g$, by Lemma 2.2,

$$\begin{aligned} \Gamma_g^* 1 + \frac{2}{n} Q_g g &= \left(\frac{3}{2(n-1)} - \frac{2}{n} \right) g \Delta \text{tr} S + \frac{2}{n-2} (\Delta S + 2 \overset{\circ}{R} m \cdot S) + \frac{n^2 - 10n + 12}{2(n-1)(n-2)} \nabla^2 \text{tr} S \\ &\quad - \frac{4}{n} |S|^2 g - \frac{n^2 - 2n + 4}{2(n-1)} (\text{tr} S) S + \left(1 - \frac{n^2 - 2n + 4}{2(n-1)(n-2)} \right) (\text{tr} S)^2 g \\ &= - \frac{n-4}{2n(n-1)} g \Delta \text{tr} S + \frac{2}{n-2} (\Delta S + 2 \overset{\circ}{R} m \cdot S) + \frac{n^2 - 10n + 12}{2(n-1)(n-2)} \nabla^2 \text{tr} S \\ &\quad - \frac{4}{n} |S|^2 g - \frac{n^2 - 2n + 4}{2(n-1)} (\text{tr} S) S + \frac{n(n-4)}{2(n-1)(n-2)} (\text{tr} S)^2 g. \end{aligned}$$

Applying Proposition 2.1,

$$\begin{aligned} \Gamma_g^* 1 + \frac{2}{n} Q_g g &= \frac{2}{n-2} B_g - \frac{n-4}{2n(n-1)} g \Delta \text{tr} S + \left(\frac{2}{n-2} + \frac{n^2 - 10n + 12}{2(n-1)(n-2)} \right) \nabla^2 \text{tr} S \\ &\quad + \frac{2(n-4)}{n-2} S \times S + \left(\frac{2}{n-2} - \frac{4}{n} \right) |S|^2 g + \left(\frac{4}{n-2} - \frac{n^2 - 2n + 4}{2(n-1)} \right) (\text{tr} S) S \\ &\quad + \frac{n(n-4)}{2(n-1)(n-2)} (\text{tr} S)^2 g. \end{aligned}$$

That is,

$$\begin{aligned}
\Gamma_g^* 1 + \frac{2}{n} Q_g g &= \frac{2}{n-2} B_g - \frac{n-4}{2n(n-1)} g \Delta \text{tr} S + \frac{n-4}{2(n-1)} \nabla^2 \text{tr} S + \frac{2(n-4)}{n-2} S \times S \\
&\quad - \frac{2(n-4)}{n(n-2)} |S|^2 g - \frac{n^2(n-4)}{2(n-1)(n-2)} (\text{tr} S) S + \frac{n(n-4)}{2(n-1)(n-2)} (\text{tr} S)^2 g \\
&= \frac{2}{n-2} B_g + \frac{n-4}{2(n-1)} \left(\nabla^2 \text{tr} S - \frac{1}{n} g \Delta \text{tr} S \right) + \frac{2(n-4)}{n-2} \left(S \times S - \frac{1}{n} |S|^2 g \right) \\
&\quad - \frac{n^2(n-4)}{2(n-1)(n-2)} (\text{tr} S) \left(S - \frac{1}{n} (\text{tr} S) g \right) \\
&= \frac{2}{n-2} B_g + \frac{n-4}{2(n-1)(n-2)} T_g,
\end{aligned}$$

where

$$T_g := (n-2) \left(\nabla^2 \text{tr}_g S_g - \frac{1}{n} g \Delta_g \text{tr}_g S_g \right) + 4(n-1) \left(S_g \times S_g - \frac{1}{n} |S_g|^2 g \right) - n^2 (\text{tr}_g S_g) \mathring{S}_g.$$

Therefore,

$$J_g = -\frac{1}{2} \Gamma_g^* 1 = \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g.$$

□

Immediately, we have the following generalization of Theorem 1.2:

Corollary 2.5. *Let (M^n, g) be a Q -singular n -dimensional Riemannian manifold. Then $1 \in \ker \Gamma_g^*$ if and only if (M^n, g) is J -flat or equivalently (M^n, g) satisfies*

$$(2.11) \quad B_g = -\frac{n-4}{4(n-1)} T_g$$

with vanishing Q -curvature.

Remark 2.6. Similar result holds for Ricci curvature: a vacuum static space admits a constant static potential if and only if it is Ricci flat (cf. [8]).

3. AN ALMOST-SCHUR LEMMA FOR Q -CURVATURE

Since the tensor J_g can be interpreted as a higher-order analogue of Ricci tensor, we can also derive the Schur lemma for J_g as follows:

Theorem 3.1 (Schur lemma). *Let (M^n, g) be an n -dimensional J -Einstein manifold with $n \neq 4$ or equivalently,*

$$B_g = -\frac{n-4}{4(n-1)} T_g,$$

then Q_g is a constant on M .

Proof. By the assumption, $J_g = \Lambda g$ for some smooth function Λ on M . Then

$$\Lambda = \frac{1}{n} \text{tr}_g J_g = \frac{1}{n} Q_g$$

and

$$d\Lambda = \text{div}_g J_g = \frac{1}{4} dQ_g.$$

Therefore,

$$\frac{n-4}{4n} dQ_g = 0$$

on M , which implies that Q_g is a constant on M provided $n \neq 4$.

□

Remark 3.2. When $n = 4$, J -Einstein metrics are exactly Bach flat ones. Due to the conformal invariance of Bach flatness in dimension 4, we can easily see that the constancy of Q -curvature can not always be achieved. Thus the above Schur Lemma does not hold for 4-dimensional manifolds which is exactly like the classic Schur Lemma for surfaces.

In fact, a more general result can be derived:

Theorem 3.3 (Almost-Schur Lemma). *For $n \neq 4$, let (M^n, g) be an n -dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$(3.1) \quad \int_M (Q_g - \overline{Q}_g)^2 dv_g \leq \frac{16n(n-1)}{(n-4)^2} \int_M |\overset{\circ}{J}_g|^2 dv_g,$$

where \overline{Q}_g is the average of Q_g . Moreover, the equality holds if and only if (M^n, g) is J -Einstein.

The proof is along the same line as in [7]. For completeness, we include it here for the convenience of readers. For more details, please refer to [7].

Proof. Let u be the unique solution to

$$\begin{cases} \Delta_g u = Q_g - \overline{Q}_g, \\ \int_M u dv_g = 0. \end{cases}$$

Then

$$\int_M (Q_g - \overline{Q}_g)^2 dv_g = \int_M (Q_g - \overline{Q}_g) \Delta_g u dv_g = - \int_M \langle \nabla Q_g, \nabla u \rangle dv_g = - \frac{4n}{n-4} \int_M \langle \text{div}_g \overset{\circ}{J}_g, \nabla u \rangle,$$

where for the last step we use the fact

$$\text{div}_g \overset{\circ}{J}_g = \text{div}_g \left(J_g - \frac{1}{n} Q_g g \right) = \frac{1}{4} dQ_g - \frac{1}{n} dQ_g = \frac{n-4}{4n} dQ_g.$$

Integrating by parts,

$$\begin{aligned}
-\frac{4n}{n-4} \int_M \langle \text{div}_g \overset{\circ}{J}_g, \nabla u \rangle dv_g &= \frac{4n}{n-4} \int_M \langle \overset{\circ}{J}_g, \nabla^2 u \rangle dv_g \\
&= \frac{4n}{n-4} \int_M \langle \overset{\circ}{J}_g, \nabla^2 u - \frac{1}{n} g \Delta_g u \rangle dv_g \\
&\leq \frac{4n}{n-4} \left(\int_M |\overset{\circ}{J}_g|^2 dv_g \right)^{\frac{1}{2}} \left(\int_M \left| \nabla^2 u - \frac{1}{n} g \Delta_g u \right|^2 dv_g \right)^{\frac{1}{2}} \\
&= \frac{4n}{n-4} \left(\int_M |\overset{\circ}{J}_g|^2 dv_g \right)^{\frac{1}{2}} \left(\int_M |\nabla^2 u|^2 - \frac{1}{n} (\Delta_g u)^2 dv_g \right)^{\frac{1}{2}}.
\end{aligned}$$

From *Bochner formula* and the assumption $\text{Ric}_g > 0$,

$$\int_M |\nabla^2 u|^2 dv_g = \int_M (\Delta_g u)^2 dv_g - \int_M \text{Ric}_g(\nabla u, \nabla u) dv_g \leq \int_M (\Delta_g u)^2 dv_g.$$

Thus,

$$\begin{aligned}
\int_M (Q_g - \overline{Q}_g)^2 dv_g &\leq \frac{4n}{n-4} \left(\int_M |\overset{\circ}{J}_g|^2 dv_g \right)^{\frac{1}{2}} \left(\frac{n-1}{n} (\Delta_g u)^2 dv_g \right)^{\frac{1}{2}} \\
&= \frac{4n}{n-4} \left(\int_M |\overset{\circ}{J}_g|^2 dv_g \right)^{\frac{1}{2}} \left(\frac{n-1}{n} (Q_g - \overline{Q}_g)^2 dv_g \right)^{\frac{1}{2}}.
\end{aligned}$$

That is,

$$\int_M (Q_g - \overline{Q}_g)^2 dv_g \leq \frac{16n(n-1)}{(n-4)^2} \int_M |\overset{\circ}{J}_g|^2 dv_g.$$

Now we consider the equality case.

If g is J -Einstein, then Q_g is a constant by *Schur Lemma* (Theorem 1.8). Thus both sides of inequality (3.1) vanish and equality is achieved.

On the contrary, assume in (3.1) equality is achieved:

$$\int_M (Q_g - \overline{Q}_g)^2 dv_g = \frac{16n(n-1)}{(n-4)^2} \int_M |\overset{\circ}{J}_g|^2 dv_g.$$

Then in particular we have

$$\text{Ric}(\nabla u, \nabla u) = 0,$$

which implies that $\nabla u = 0$ and hence u is a constant on M , since we assume $\text{Ric}_g > 0$.

Thus $Q \equiv \overline{Q}$ on M and

$$\int_M |\overset{\circ}{J}_g|^2 dv_g = \frac{(n-4)^2}{16n(n-1)} \int_M (Q_g - \overline{Q}_g)^2 dv_g = 0.$$

Therefore, $\overset{\circ}{J}_g \equiv 0$ on M , i.e. (M, g) is J -Einstein. □

Remark 3.4. By assuming

$$\text{Ric} \geq -(n-1)Kg$$

for some constant $K \geq 0$ and following the proof in [5], the inequality (3.1) can be improved to

$$(3.2) \quad \int_M (Q_g - \overline{Q}_g)^2 dv_g \leq \frac{16n(n-1)}{(n-4)^2} \left(1 + \frac{nK}{\lambda_1}\right) \int_M |\overset{\circ}{J}_g|^2 dv_g,$$

where $\lambda_1 > 0$ is the first non-zero eigenvalue of $(-\Delta_g)$.

Now we can derive an equivalent form of inequality (3.1):

Corollary 3.5. *For $n \neq 4$, let (M^n, g) be an n -dimensional closed Riemannian manifold with positive Ricci curvature. Then*

$$(3.3) \quad (\text{Vol}_g(M))^{-\frac{n-8}{n}} \int_M \sigma_2^J(g) dv_g \leq \frac{n-1}{2n} Y_Q^2(g).$$

Moreover, the equality holds if and only if (M^n, g) is J -Einstein.

Proof. Note that

$$\sigma_1^J(g) = \text{tr}_g S_J = \frac{1}{4(n-1)} Q_g$$

and

$$\sigma_2^J(g) = \frac{1}{2} ((\sigma_1^J)^2 - |S_J|^2) = \frac{n-1}{2n} (\sigma_1^J)^2 - \frac{1}{2(n-4)^2} |\overset{\circ}{J}_g|^2,$$

where we use the fact

$$|S_J|^2 = \left| \overset{\circ}{S}_J + \frac{1}{n} (\text{tr}_g S_J) g \right|^2 = \left| \frac{1}{n-4} \overset{\circ}{J}_g + \frac{1}{n} (\sigma_1^J) g \right|^2 = \frac{1}{(n-4)^2} |\overset{\circ}{J}_g|^2 + \frac{1}{n} (\sigma_1^J)^2.$$

By substituting these terms in the inequality (3.1), we get

$$\left(\int_M \sigma_1^J(g) dv_g \right)^2 \geq \frac{2n}{n-1} \text{Vol}_g(M) \int_M \sigma_2^J(g) dv_g.$$

Therefore,

$$\begin{aligned} \int_M \sigma_2^J(g) dv_g &\leq \frac{n-1}{2n} (\text{Vol}_g(M))^{-1} \left(\int_M \sigma_1^J(g) dv_g \right)^2 \\ &= \frac{n-1}{2n} (\text{Vol}_g(M))^{\frac{n-8}{n}} \left(\frac{\int_M \sigma_1^J(g) dv_g}{(\text{Vol}_g(M))^{\frac{n-4}{n}}} \right)^2 \\ &= \frac{n-1}{2n} (\text{Vol}_g(M))^{\frac{n-8}{n}} Y_Q^2(g). \end{aligned}$$

□

Remark 3.6. Note that, the Q -Yamabe quotient

$$Y_Q(g) := \frac{\int_M \sigma_1^J(g) dv_g}{(Vol_g(M))^{\frac{n-4}{n}}}$$

is scaling invariant and in particular, when $n = 8$,

$$\int_M \sigma_2^J(g) dv_g \leq \frac{7}{16} Y_Q^2(g),$$

provided that $Ric_g > 0$, where the equality holds if and only if (M, g) is J -Einstein.

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